P 6 Adaptive anisotropic boundary element methods (Th. Apel, O. Steinbach) → AO, NS

The focus in this project is on the numerical analysis of adaptive boundary element methods for the numerical solution of elliptic second order partial differential equations with boundary conditions of different types. In the case of a polyhedral domain it is well known that the solution has special singular forms at corners and edges, see, e.g., [8] for a priori error estimates. Hence, to retain an optimal order of convergence of the related Galerkin boundary element solution, graded meshes can be used which can be generated by using information from a posteriori error estimators. In addition to the development and analysis of appropriate a posteriori error estimators we need to establish related approximation properties of anisotropic boundary element spaces in fractional Sobolev spaces, in particular in $H^{\pm 1/2}(\Gamma)$. Moreover, the implementation of anisotropic boundary element methods requires a stable evaluation of the involved surface integrals by means of numerical and semi–analytic quadrature formulae. An efficient and accurate boundary element realization of the Dirichlet to Neumann map can be used for the solution of related optimal control problems, shape optimization problems, and within domain decomposition methods, for the coupling with finite element methods.

State of the art. Boundary integral equation and fast boundary element methods [4, 10, 9, 13] are well established, in particular when considering partial differential equations with piecewise constant coefficients, and when considering exterior boundary value problems. Adaptive boundary element methods based on suitable a posteriori error estimators are mainly considered for two–dimensional problems so far, there are only a few contributions on particular three–dimensional problems, see, e.g., [2, 5]. While the analysis of anisotropic finite element methods is well established [1], almost nothing is known on approximation properties of anisotropic boundary element spaces in fractional Sobolev spaces, in particular in $H^{\pm 1/2}(\Gamma)$. For the adaptive design of graded meshes appropriate information from a posteriori error estimators is required. Besides $h - h/2$ type error estimators [3] we will consider techniques which are based on complementary boundary integral equations [11]. In both cases representatives of the error function are computed from which we can derive appropriate information to drive an anisotropic mesh refinement. Boundary element methods also allow for an accurate and efficient approximation of Steklov–Poincaré operator as used in many applications. For the solution of boundary control problems by using boundary integral equation techniques, see, e.g., [7], and for domain decomposition methods, e.g., [12]. In particular for a non–symmetric boundary element approximation of the Steklov–Poincaré operator, integration techniques and stability conditions as used in the analysis of mortar finite element domain decomposition methods [6] are required. But in most cases, the same meshes are used to approximate the Cauchy data representing different physical quantities.

Thesis project to be supervised by O. Steinbach. As a model problem we first consider a Dirichlet boundary value problem in a polyhedral bounded domain $\Omega \subset \mathbb{R}^3$. To find the unknown Neumann datum the direct approach with single and double layer potentials is applied which results in a weakly singular first kind boundary integral equation to be solved. To estimate the error of the Galerkin boundary element solution we consider a complementary second kind boundary integral equation of the adjoint double layer potential from which we obtain more detailed information on the error distribution. This will be the basis for the design of graded meshes to define appropriate boundary element spaces. As in finite element methods we first analyze approximation properties of lowest order boundary element spaces in $L_2(\Gamma)$. Based on these results we will also consider a priori error estimates in $H^{\pm 1/2}(\Gamma)$. For the localization of the involved Sobolev norms we will use multilevel techniques in combination with fast boundary element methods. In particular we will apply the fast multipole method to accelerate both the direct boundary element solution, and the a posteriori error estimator. To end up with a most flexible approach we will use different boundary element meshes to define approximations of the given Dirichlet datum, and of the unknown Neumann datum. But this requires the evaluation of regular and singular surface integrals containing basis functions which are defined with respect to different boundary element meshes. The integration will be done by using numerical and semi–analytic quadrature formulae. Moreover, when considering the non–symmetric boundary element approximation of the Steklov–Poincaré operator this requires an appropriate stability condition to be satisfied, which is related to the stability of mortar finite element domain decomposition methods.

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