The Gradient Discretisation Method: framework and tools for the numerical analysis of elliptic and parabolic models

The gradient discretisation method (GDM) is a generic framework for the design and convergence analysis of numerical schemes for elliptic and parabolic PDEs [1]. The GDM is built on the choice of a few discrete elements $\mathcal{D} = (X_{\mathcal{D}}, \nabla_{\mathcal{D}}, \Pi_{\mathcal{D}})$, together called a 'gradient discretisation' (GD):

- a finite dimensional space $X_{\mathcal{D}}$,
- a gradient reconstruction $\nabla_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Omega)^d$,
- a function reconstruction $\Pi_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Omega)$.

Substituting, in lieu of the corresponding continuous space and operators, these discrete elements in the weak formulation of the PDE gives rise to a numerical scheme called the 'gradient scheme' (GS). Considering for example the simple diffusion model $-\mathrm{div}(\Lambda\nabla u) = f$ with homogeneous Dirichlet boundary conditions on $\Omega \subset \mathbb{R}^d$, whose weak formulation is

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega \Lambda\nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega),$$

the corresponding GS reads

$$\text{Find } u_{\mathcal{D}} \in X_{\mathcal{D}} \text{ such that } \int_\Omega \Lambda\nabla_{\mathcal{D}} u_{\mathcal{D}} \cdot \nabla_{\mathcal{D}} v_{\mathcal{D}} = \int_\Omega f\Pi_{\mathcal{D}} v \quad \forall v_{\mathcal{D}} \in X_{\mathcal{D}}.$$

Three to five properties only (depending on the model) on sequences $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of GDs ensure that the corresponding GSs converge for a wide range of models, from linear elliptic and parabolic PDEs to non-linear models including p-Laplace equations, the Stokes and Navier–Stokes equations, degenerate parabolic equations, and optimal control problems with elliptic state equations.

Specific choices of GDs correspond to specific schemes, and the GDM analysis therefore seamlessly applies to many numerical methods, from finite elements (conforming, non-conforming and mixed, including mass-lumped versions), finite volume methods, to virtual element methods, etc.

In this talk, I will give an overview of the GDM, the numerical methods it encompasses, and the results it yields. I will also cover some generic compactness results developed alongside the GDM to facilitate the convergence analysis for non-linear time-dependent problems. Some of the noticeable results stemming from the GDM include a response to a long-standing conjecture on the super-convergence of the Two-Point Flux Approximation finite volume method (popular in petroleum engineering), and a novel uniform-in-time convergence result for degenerate parabolic equations.

References

[1] Jérôme Droniou, Robert Eymard, Thierry Gallouët, Cindy Guichard, and Raphaèle Herbin. *The gradient discretisation method.* 511p, 2018. Mathematics & Applications, Springer, Heidelberg. To appear. URL: https://hal.archives-ouvertes.fr/hal-01382358.